

Guess the Larger Number

GNEDIN, A

This work is licensed under a Creative Commons Attribution 3.0 License.

For additional information about this publication click this link.

<http://qmro.qmul.ac.uk/xmlui/handle/123456789/16207>

Information about this research object was correct at the time of download; we occasionally make corrections to records, please therefore check the published record when citing. For more information contact scholarlycommunications@qmul.ac.uk

ALEXANDER GNEDIN (London)

Guess the Larger Number

B *I myself have invented a game. Well, think of a number.*

A *I got a number.*

B *Me too. Now, tell me yours.*

A *Seven.*

B *Seven. Mine is eight – I won.*

Sergey Solovoyov, *Assa (conversation of Bananan and Alike)*

Abstract We discuss variations of the zero-sum game where Bob selects two distinct numbers, and Alice learns one of them to make a guess which of the numbers is the larger.

2010 Mathematics Subject Classification: 91A60, 60G09, 60G40, 62L15.

Key words and phrases: Cover's problem, zero-sum game, dominance, secretary problem, exchangeability, probability inequalities, admissibility.

1. Introduction. To play a guessing game, Bob takes two cards and on each writes a different number. Alice randomly chooses one of these cards and peeks at the number. She must then decide which of the numbers is the larger: the number in her hand or the other, concealed number. She wins each time her guess is correct.

By blind guessing Alice wins with probability $1/2$. Interestingly, there is a better strategy. The idea is to compare the observed number x with a threshold number t , sampled from some probability distribution, for instance the normal. If $x \geq t$ the strategy accepts x as the larger number of two on the cards, otherwise rejects x and decides that the concealed number is the larger. By symmetry, Alice is right or wrong with the same probability, if t falls either above or below *both* Bob's numbers. But if t falls *between* the two numbers, the decision is correct for sure: the observed x will be accepted if it is the larger number and rejected otherwise. The advantage of the strategy is due to the latter possibility, which has nonzero probability whichever Bob's numbers happen to be.

For many people this appears counter-intuitive, since a random choice from two unknown numbers seems to bear no information on how the numbers

compare to one another. The history of the paradox can be traced back to the work on theoretical statistics by David Blackwell [2] and Bruce Hill [17]. Much of the interest was sparked by Tom Cover's [8] half-page abstract, where the game framework was explicitly introduced. Steve Samuels [26] noticed that the game belongs to the circle of questions around Martin Gardner's *Game of Googol* and *Secretary Problem*, where the guessing player aims to stop at the maximum of a sequence of unknown numbers observed in random order [1, 10, 27]. The conundrum was popularised by Thomas Bruss [5] and Ted Hill [18], and many further fascinating connections with the classical probability problems were subsequently found [4, 11, 12, 14, 15, 19, 25, 31].

From the perspective of the zero-sum game theory the guessing game is fair, and the blind guessing is an equalising minimax strategy. The advantage of a threshold strategy over blind guessing is an interesting dominance phenomenon that cannot occur in games with finitely many pure strategies. Bob has no minimax strategy, and so the game has no solution although has a definite value. The latter means that Bob is capable of keeping Alice's advantage arbitrarily small.

A common fallacy associated with the game is that a smart strategy for Bob were choosing the numbers close to one another. This recipe is good to keep the advantage of a randomised threshold strategy small, but may fail when Alice uses some other strategy. For instance, if Bob employs a pure game strategy involving any two given numbers, then Alice can respond with a sure-fire guessing. Much more importantly, Bob should really take care of that the odds given the observed number are about equal. Such epsilon-minimax strategies for the number-writing player have been designed in [1, 10–12].

In this paper we discuss strategies for both players, thus giving a complete account of the game. In particular, we will characterise all Alice's minimax strategies, that guarantee the winning probability $1/2$. We shall also consider variations of the game by granting the number-writing player different powers, to stress the roles of the admissible number range and exchangeability implied by random choice of a card. On this way we will introduce a more involved game with two piles of cards. In the new game Bob takes, let's say, 52 cards and on each writes a different number. The cards are turned face down, shuffled and dealt in two piles of given sizes. Alice collects the first pile, opens the cards and guesses if the number largest of all 52 numbers is in her hand or in the second pile. This can be regarded as a version of the Game of Googol, where the stopping player has some return options. We will see that properties of the two-pile game largely depend on whether the deck is divided evenly or not. If Alice gets exactly half of the deck, she can take some advantage from learning the numbers by employing a threshold strategy, just like in the game with two cards. If the piles are of different size, Bob has a smart way to generate the numbers, such that even if Alice finds out his algorithm, she cannot do better than with the obvious strategy of betting on

the bigger pile.

Although the paper is mostly self-contained, we shall assume the reader is familiar with random variables and probability distributions on the level of a ‘second course’ in probability or statistics. For fundamentals of the game-theoretic approach to statistical decision theory we refer to the classic text [3].

2. The general setting Before embarking on a discussion of the paradox it is helpful to consider a wider framework for the guessing games. Suppose that the revealed number x and the concealed number y are sample values of two random variables X and Y , respectively. We assume $P(X = Y) = 0$, that probability of a ‘tie’ is zero, i.e. the numbers are different and so one of them is always the larger and the other is the smaller. The *joint* probability distribution of the pair X, Y is a strategy for Bob. The rules of a particular game specify, sometimes implicitly, a class of joint distributions that Bob is allowed to use. The rules of the game belong to the common knowledge of players.

Having observed a value x , Alice is willing to guess which of the numbers is the larger. If she decides by tossing a fair coin, she is right with probability $1/2$. Whether there are better strategies depends on the rules of the game.

A pure strategy for Alice will be identified with a set $D \subset \mathbb{R}$ called a *decision set*. According to strategy D , the observed number x is accepted as the larger if and only if $x \in D$. To be able to define probabilities associated with the strategy we will assume that the set D is Borel.

Alice has two pure *blind guessing* strategies: one with $D = \mathbb{R}$ always accepting x , and the other with $D = \emptyset$ always rejecting x . A pure *threshold strategy* has $D = [t, \infty)$ with some fixed threshold t .

EXAMPLE 2.1 It is easy to find a solution to the unconstrained game, where the joint distribution for X, Y can be arbitrary. Then playing $X = 0$ and $Y = \pm 1$ with equal probabilities, Bob achieves that Alice wins with probability $1/2$ no matter how she decides. Thus $1/2$ is the value of this game.

EXAMPLE 2.2 The game is more interesting when Bob is in some way restricted in choosing the joint distribution for X, Y . In the arrangement game introduced in [14] he is *given* two numbers sampled independently from the uniform distribution on $[0, 1]$, but may choose which of the numbers is shown to Alice. The property characterising the relevant class of distributions is that the pair X, Y arranged in increasing order has uniform distribution on $\{(a, b) : 0 < a < b < 1\}$. Still, Bob has an equalising strategy: show the number which is closer to $1/2$. By symmetry of the uniform distribution about the median, the hidden number is equally likely to be bigger or smaller than the shown number.

In line with the paradigm of the zero-sum game theory, when choosing a strategy Bob should expect Alice's response most favourable for her. The best response is an optimal strategy in a Bayesian decision problem, where Alice knows exactly the joint distribution of X, Y and can unambiguously evaluate strategies to find the one with the highest winning probability. Having observed x , she will be able to calculate the conditional probability of winning by accepting the number in hand, that is

$$\pi(x) := P(X > Y | X = x) = P(Y \leq x | X = x).$$

If she plays pure strategy D , the total probability of the correct guess is equal to

$$\begin{aligned} P(X > Y, X \in D) + P(X < Y, X \in D^c) &= E[\pi(X) \mathbb{1}_{\{X \in D\}} \\ &\quad + (1 - \pi(X)) \mathbb{1}_{\{X \in D^c\}}], \end{aligned}$$

where $\mathbb{1}_{\{\dots\}}$ equals 1 if \dots is true and equals 0 otherwise, D^c is the complement of set D and E denotes the expectation. The random variable $\pi(X)$ is obtained by substituting X in the function $\pi : \mathbb{R} \rightarrow [0, 1]$, so the distribution of $\pi(X)$ is determined by the joint distribution of X, Y . A minute reflection shows that the total probability of the correct guess will be maximised if Alice accepts x when $\pi(x) > 1/2$, and rejects x when $\pi(x) < 1/2$; while decision in the case $\pi(x) = 1/2$ can be arbitrary¹. Playing so, when $X = x$ she will be right with probability

$$\max(\pi(x), 1 - \pi(x)) = \frac{1}{2} + \left| \pi(x) - \frac{1}{2} \right|,$$

hence the overall probability of the correct guess becomes

$$v := \frac{1}{2} + E \left| \pi(X) - \frac{1}{2} \right|. \quad (1)$$

Note that the maximal probability v is achieved by a pure strategy $D = \{x : \pi(x) \geq 1/2\}$. But if $\pi(X)$ assumes value $1/2$ with positive probability, there are also mixed strategies, with randomised decision set, that achieve (1).

A useful related concept is the median of the conditional distribution of Y given that $X = x$

$$\mu(x) = \inf \left\{ y : P(Y \leq y | X = x) \geq \frac{1}{2} \right\}.$$

In view of inequality

$$P(Y \leq \mu(x) | X = x) \geq \frac{1}{2}$$

¹In fact, optimality of this strategy is a consequence of the principle of *dynamic programming*.

the optimal decision set can be determined by the condition

$$x \geq \mu(x). \quad (2)$$

EXAMPLE 2.3 Here is an important example. Suppose X and Y are independent and identically distributed (iid) with some known continuous distribution. The Bayesian decision problem is then the simplest case of the *Full-Information Secretary Problem* [27]. By independence $\pi(x) = P(Y \leq x) = P(X \leq x)$ is the common cumulative distribution function, and the random variable $\pi(X)$ is uniformly distributed on $[0, 1]$ (the fact known as the probability integral transform). Since $|\pi(X) - 1/2|$ is uniformly distributed on $[0, 1/2]$, formula (1) readily yields $v = 3/4$. The conditional median is a constant μ obtained as the minimal solution to the equation $\pi(t) = 1/2$. Alice's optimal strategy is of the threshold type, with threshold coinciding with the median.

3. The paradox of two cards

3.1. Threshold strategies We turn now to Tom Cover's game, where Alice picks a card at random. This has the effect of *exchangeability* [7], meaning that permutation Y, X has the same joint distribution as X, Y . In terms of the bivariate distribution function, exchangeability holds if

$$P(X \leq x, Y \leq y) = P(Y \leq x, X \leq y)$$

for all x, y . In particular, the random variables X and Y are identically distributed and $P(X > Y) = P(Y > X) = 1/2$. A pure strategy for Bob in this game can be modeled as a pair of fixed numbers a, b with $a < b$, so that the associated exchangeable variables X, Y are these numbers arranged in random order.

We note in passing that if X and Y are iid (as in Example 2.3) then they are exchangeable, but exchangeability and independence are very different properties. Thus for iid X, Y the 'no tie' condition $P(X = Y) = 0$ means that $P(X = x) = 0$ for every x , that is the distribution of X is continuous. In contrast to that, exchangeability does not rule out discrete distributions, e.g. for each Bob's pure strategy the range of X has two elements.

Suppose Alice employs a randomisation device to generate a sample value t of random variable T . To exclude cooperation between the players, as inherent to the zero-sum game, we shall assume that T is independent of X, Y . The associated threshold strategy accepts x if $x \geq t$, and rejects x otherwise. By independence and exchangeability the triple T, X, Y has the same joint

distribution as T, Y, X , thus

$$\begin{aligned}
 P(X \geq T, Y \geq T, X > Y) &= P(X \geq T, Y \geq T, Y > X) \\
 &= \frac{1}{2}P(X \geq T, Y \geq T), \\
 P(X < T, Y < T, Y > X) &= \frac{1}{2}P(X < T, Y < T), \\
 P(X < T \leq Y) &= P(Y < T \leq X) \\
 &= \frac{1}{2}P(X < T \leq Y) + \frac{1}{2}P(Y < T \leq X).
 \end{aligned}$$

Summing probabilities on the right-hand side yields $1/2$, hence adding to the sum $P(Y < T \leq X)$ the probability of correct guess with the threshold strategy becomes

$$w := \frac{1}{2} + P(Y < T \leq X). \quad (3)$$

Strict inequality $w > 1/2$ is ensured (for arbitrary distribution of X, Y) if T is *fully supported*, i.e. with cumulative distribution function

$$F(t) = P(T \leq t) \quad (4)$$

strictly increasing for $t \in \mathbb{R}$. The latter condition is equivalent to

$$P(a < T \leq b) = F(b) - F(a) > 0$$

for all $a < b$, which is true whenever T has everywhere positive density.

Observe that formula (3) is valid under a weaker assumption that T, X, Y have the same joint distribution as T, Y, X (that is, X, Y are conditionally exchangeable given a value of T). The case where all three variables T, X, Y are exchangeable suggests the following variation of the game, in the spirit of Secretary Problem with a ‘training sample’ [6].

EXAMPLE 3.1 For readers unfamiliar with the Secretary Problem this application of (3) might appear even more surprising than the paradox of two cards. Suppose Bob writes three distinct numbers on separate cards, which are shuffled and dealt face down in a row. Alice turns face up the first and the second card, and must then guess if the number on the second card is larger than a concealed number on the third card. Using the number on the first card as a threshold to compare with the number on the second card, Alice wins with probability $w = 1/2 + P(Y < T < X) = 1/2 + 1/6 = 2/3$, because due to exchangeability all $3! = 6$ rankings of T, X, Y are equally likely. This example stresses the role of exchangeability in affecting the odds. Although three numbers are a priori unknown, Alice can substantially benefit from comparing the first two.

For any given joint distribution of X, Y the best of threshold strategies has nonrandom threshold found by maximising $P(Y < t \leq X)$. To make this probability positive, t must lie between supremum and infimum of the range of X , although for some such t the probability can be zero². For instance, if Bob plays a mixture of pure strategies 1, 2 and 7, 8 we will have $P(Y < t \leq X) = 0$ for $t = 5$. If X and Y are not iid, typically the optimal threshold does not coincide with the median of X .

3.2. A probability inequality Once the first surprise is over, one might attempt explaining the paradox by arguing that the bigger the observed number x , the more likely it is the larger. However, exchangeability beats intuition also in this respect. Contrary to the iid case, in general the conditional probability $\pi(x)$ is neither increasing in x , nor even exceeds $1/2$ for all sufficiently big x . As a consequence, the optimal Alice's counter-strategy against X, Y with given distribution may not be of the threshold type.

EXAMPLE 3.2 Here is an example of erratic behaviour. For every integer j there is a pure strategy for Bob with numbers $3j, 3j + 1$. Take a mixture of these pure strategies with some positive weights, that is pick j from any probability distribution on the set of integers, then arrange the numbers $3j$ and $3j + 1$ in random order. For X, Y thus defined, $\pi(x) = 0$ for x a multiple of 3, but $\pi(x) = 1$ for $x = 1$ modulo 3. Obviously, Alice's best response is not a threshold strategy, rather a sure-fire strategy deciding that $x > y$ whenever $x = 1$ modulo 3, so with decision set $D = \{3j + 1 : j \in \mathbb{Z}\}$.

A similar example illustrates that $P(X > Y | X \geq t)$ need not be increasing in t . What *is* the correct intuition, is that the message 'the observed number exceeds a threshold' increases the likelihood of the event $X > Y$, because X is certainly the larger when the hidden number falls below the threshold, whereas the variables are indistinguishable otherwise. Likewise, the message 'the observed number falls below a threshold' speaks against $X > Y$ because the hidden number may still exceed the threshold. Formally, this conditional probability in focus is at least $1/2$, which is precisely the inequality underlying the paradox. We will show this for a random threshold T independent of

²The assertion on top of p. 290 in [26] needs to be corrected.

exchangeable X and Y . Indeed,

$$\begin{aligned}
 P(X > Y, X \geq T) &= P(X > Y, X \geq T, Y \geq T) + P(X > Y, X \geq T, Y < T) \\
 &= \frac{1}{2}P(X \geq T, Y \geq T) + P(Y < T \leq X) \\
 &= \frac{1}{2}\{P(X \geq T, Y \geq T) + P(Y < T \leq X)\} \\
 &\quad + \frac{1}{2}P(Y < T \leq X) \\
 &= \frac{1}{2}P(X \geq T) + \frac{1}{2}P(Y < T \leq X).
 \end{aligned}$$

Assuming that $P(X \geq T) > 0$ we arrive at

$$P(X > Y | X \geq T) = \frac{1}{2} + \frac{1}{2}P(Y < T | X \geq T) \geq \frac{1}{2}, \quad (5)$$

where the bound is strict if $P(Y < T \leq X) > 0$.

To conclude, given the observed number exceeds a threshold, betting that this number is the larger is, overall, favourable. However, for a particular x , no matter how large, the bet may turn unfavourable.

Similarly, a negative outcome of the threshold comparison reduces the likelihood

$$P(X > Y | X < T) = \frac{1}{2} - \frac{1}{2}P(Y \geq T | X < T) \leq \frac{1}{2}.$$

This obvious counterpart of (5) will be useful in what follows to introduce deformations of threshold strategy.

There are several variations on the theme how comparison with a threshold affects ranking. One generalisation of (5) is the following. Let X_1, \dots, X_n be exchangeable random variables without ties. The rank R of X_1 is defined as the number of X_j 's not greater than X_1 . By exchangeability the distribution of R is uniform on n integers. Now, it can be shown that for T independent threshold, condition $X_1 > T$ increases the probability of $R > k$ for every $k = 1, \dots, n-1$. That is to say, the information that a number chosen at random out of n numbers exceeds a threshold stochastically increases the rank of this number. See [15] for applications of relations akin to (5) in optimal stopping.

3.3. T. Hill's finite game In the wonderful expository article [18] Ted Hill invites the reader to solve a guessing game where Bob is only allowed to use integer numbers from 1 to, let us say, $m+1$. This game reduces to a finite game, i.e. with finite sets of pure strategies for both players, because for Alice it is sufficient to use decision sets $D \subset \{1, \dots, m+1\}$. For the finite game

the celebrated von Neumann's Minimax Theorem ensures that the game has a solution³.

Define a mixed strategy for Bob by choosing β uniformly from the set of integers $1, 2, \dots, m$. Given β , the variables X, Y are the numbers $\beta, \beta + 1$ arranged in random order. One can think of sampling a coin from a bag of m coins with labels $\beta, \beta + 1$ on the sides, then tossing the coin and noting a number x on the upper side. The resulting joint distribution of X, Y is uniform on the set of $2m$ integer pairs

$$\{(\beta, \beta + 1) : 1 \leq \beta \leq m\} \cup \{(\beta + 1, \beta) : 1 \leq \beta \leq m\}. \quad (6)$$

When the observed number is some $1 < x < m + 1$ both options $\beta = x$ and $\beta = x - 1$ are equally likely, that is

$$P(X = x) = P(X = x, Y = x + 1) + P(X = x, Y = x - 1) = \frac{1}{m}$$

and

$$\pi(x) = \frac{P(X = x, Y = x - 1)}{P(X = x)} = \frac{1}{2}.$$

For the extremes we have

$$P(X = 1) = P(X = m + 1) = \frac{1}{2m}$$

and $\pi(1) = 0, \pi(m + 1) = 1$. Thus (1) gives the total probability of win by the best response

$$v = \frac{1}{2} + \frac{1}{2m} \left| 0 - \frac{1}{2} \right| + (m - 1) \cdot \frac{1}{m} \cdot 0 + \frac{1}{2m} \left| 1 - \frac{1}{2} \right| = \frac{1}{2} + \frac{1}{2m}.$$

Define a threshold strategy for Alice with T uniformly distributed over the set $\{2, \dots, m + 1\}$. Whichever distinct x, y from $\{1, \dots, m + 1\}$, the larger of them belongs to $\{2, \dots, m + 1\}$. Thus, whichever Bob's strategy the event $T = \max(X, Y)$ is independent of X, Y and has probability $1/m$. It follows that probability (3) can be estimated as

$$\begin{aligned} w &\geq \frac{1}{2} + P(Y < T = X) = \frac{1}{2} + P(Y < X, T = \max(X, Y)) = \\ &\quad \frac{1}{2} + P(Y < X)P(T = \max(X, Y)) = \frac{1}{2} + \frac{1}{2m}. \end{aligned}$$

We see that Alice can guarantee $1/2 + 1/(2m)$, while Bob has a strategy to bound her winning probability by this very value. It follows that $1/2 + 1/(2m)$

³Ted Hill provokes with a fallacious argument: 'it also seems obvious that the number-writer would never write a 1, since if [Alice turns] over a 1, [she] will always win by [not choosing 1]. But if he never writes a 1, he then would never write a 2 either since he never wrote a 1, and so on *ad absurdum*'.

is the value of the finite game, and that the described strategies of players are minimax.

Next, assuming that Bob can employ arbitrary integer numbers, we are in the situation where each player has infinitely many pure strategies, hence the Minimax Theorem cannot guarantee existence of the value. However, letting $m \rightarrow \infty$ in the finite game, it is clear that Bob can keep Alice's winning probability as close to $1/2$ as desired. On the other hand, since for the optimal Alice's counter-strategy $v \geq w > 1/2$, where w is the winning probability for some threshold strategy, Bob cannot achieve $1/2$ exactly. This implies that the game with integer numbers has value $1/2$.

But then $1/2$ is also the value in the game where Bob can use arbitrary real numbers. Bob has no minimax strategy and so the guessing game with real numbers has no solution.

In contrast to the game on integers, restricting Bob's numbers to a finite interval will not change the quantitative picture. Indeed, monotonic functions preserve the order relation hence the game with arbitrary real numbers can be transformed in equivalent game where the range of X, Y is bounded. The interval supporting X, Y can be as small as wanted.

To disprove a common belief, we notice that Bob also has good strategies where $|X - Y|$ is big. For instance, he can play a mixture of pure strategies $\beta k, \beta(k+1)$, with β sampled from the uniform distribution on $\{1, \dots, m\}$ and k fixed. Such a strategy is epsilon-minimax for k/m sufficiently small. The latter does not exclude that $k = |X - Y|$ itself is a big number, nevertheless any random threshold T will fall rarely between X and Y .

3.4. Minimax and admissible strategies Game-theoretically, both pure blind-guessing strategies with decision sets \mathbb{R} and \emptyset , respectively, are optimal (i.e. minimax) as they ensure the value of the game $1/2$. The same applies to any randomised blind guessing, e.g. deciding by tossing a fair coin. However, every pure threshold strategy $D = [t, \infty)$ is minimax as well, and dominates blind guessing. A mixed threshold strategy with fully supported T strongly dominates the blind guessing, in the sense that $P(\text{win}) > 1/2$ no matter how Bob plays. This bizarre dominance phenomenon cannot occur in finite games, where a strictly dominated strategy cannot be minimax.

In a discussion that came after the prize-winning article [5], Thomas Bruss and Tom Cover raised the following natural questions:

- (i) Is the guessing strategy that always succeeds with probability strictly greater than $1/2$ a threshold strategy?
- (ii) What is an optimal strategy? Is it of the threshold type?
- (iii) What could be an alternative to threshold comparison, to benefit from learning one of the numbers?

Question (i) is answered in negative by means of a very simple construction. With every guessing strategy one can associate a *dual* strategy, which always makes the opposite decision. When the strategy wins the dual loses and vice versa. Taking a mixture of a threshold strategy and its dual, with weights, say, $3/4$ and $1/4$ (respectively), we obtain a strategy winning with probability still greater than $1/2$. However, this example is not very exciting, because the mixture is dominated by the threshold strategy.

To answer two other questions one needs to be precise about the class of feasible guessing strategies, and especially about the very concept of randomised strategy. Once the framework is established, one should in the first instance focus on admissible (undominated) guessing strategies, i.e. those which cannot be improved in all situations, because the ‘optimal’ strategies must be among admissible whatever ‘optimality’ means. A minimax guessing strategy will be called *superminimax* if for any Bob’s play the strategy wins with probability strictly greater than $1/2$ and is admissible. In response to the Bruss-Cover questions we will give a complete characterisation of the superminimax strategies. In particular, we will show that every admissible minimax strategy, in all what concerns its performance, is equivalent to a threshold strategy, and so from the optimisation viewpoint the class of threshold strategies is sufficient. With the reservation that there are many equivalent strategies, the answer to (iii) is ‘no alternative’.

3.5. Characterising superminimax strategies Let us first recall some terminology. For \tilde{D} and D decision sets, we say that Alice’s pure strategy \tilde{D} dominates D if for every strategy of Bob the probability of win with \tilde{D} is at least as great as with D , and there exists Bob’s strategy such that the probability is strictly greater. We say that \tilde{D} strongly dominates D if the probability of win with \tilde{D} is always strictly greater than that with D . A guessing strategy is called *admissible* if no other strategy dominates it. For the purpose of checking dominance it is sufficient to compare the outcomes when Bob plays a pure strategy, because the general case follows by computing averages. To verify admissibility of Alice’s strategy one needs, in general, to make comparisons of outcomes also under mixed strategies of Bob.

The game in ‘normal form’ is specified by the outcomes when the players use pure strategies. If Alice plays $D \subset \mathbb{R}$ and Bob plays fixed $a < b$ the probability of correct guess is

$$p(a, b; D) := \begin{cases} \frac{1}{2}, & \text{if } a \in D, b \in D, \\ \frac{1}{2}, & \text{if } a \in D^c, b \in D^c, \\ 1, & \text{if } a \in D^c, b \in D, \\ 0, & \text{if } a \in D, b \in D^c. \end{cases} \quad (7)$$

where $1/2$ appears due to exchangeability of X, Y .

Although simple, the analysis of pure strategies gives some hints. Identifying D with its indicator function

$$F(x) = 1(x \in D) \quad (8)$$

the entry 0 in (7) is the situation of *inversion* in F :

$$a < b, \quad F(a) > F(b).$$

Inspecting (7) it is clear that D is minimax if and only if F has no inversions. But such D is either $D = \emptyset$ or \mathbb{R} or a right-tailed halflife, either (t, ∞) or $[t, \infty)$ for some $t \in \mathbb{R}$.

Regarding the admissibility of pure strategies, we note that if D is bounded from the above, then this strategy is dominated by $\tilde{D} = D \cup [\sup D, \infty)$, because \tilde{D} has less inversions, while other entries of (7) are the same as for D . Likewise, if D^c is bounded from below, then $\tilde{D} = D \setminus (-\infty, \inf D^c)$ dominates D . On the other hand, if $\sup D = \infty$, $\inf D^c = -\infty$ then D cannot be dominated by a pure strategy. Indeed, suppose \tilde{D} dominates D . For $a \in D^c, b \in D$ and $a < x < b$ it holds that if $x \in D$ then $1 = p(a, x; D) \leq p(a, x; \tilde{D})$ implies $p(a, x, \tilde{D}) = 1$, hence $a \in \tilde{D}^c$, $x \in \tilde{D}$. If $x \in D^c$ then $1 = p(x, b; D) \leq p(x, b, \tilde{D})$ implies $p(x, b, \tilde{D}) = 1$, and hence $x \in \tilde{D}^c, b \in \tilde{D}$. It follows that the sets D and \tilde{D} coincide between a and b , and sending $a \rightarrow -\infty, b \rightarrow \infty$ along D and D^c , respectively, we get $\tilde{D} = D$. It will be clear from what follows that this kind of argument still works when a mixed strategy is taken in the role of \tilde{D} , thus a pure strategy D with $\sup D = \infty, \inf D^c = -\infty$ is admissible.

We may now conclude that

A pure guessing strategy is admissible and minimax if and only if the decision set is either $[t, \infty)$ or (t, ∞) for some $t \in \mathbb{R}$.

Turning to randomised strategies, statistical decision theory suggests two ways to introduce this concept:

- (a) by *mixing*, that is spreading a probability measure over the set of pure strategies,
- (b) by means of a *kernel*, which for every x determines a distribution over the set of possible actions.

The relation between these two approaches has been thoroughly discussed, see e.g. [20] and references therein. In the context of the guessing game, the approach (a) specialises as a *random decision set* $\mathcal{D} \subset \mathbb{R}$. To define a kernel as in (b) we need just one function $F : \mathbb{R} \rightarrow [0, 1]$ specifying probabilities $F(x)$ and $1 - F(x)$ of two actions ‘accept x ’ and ‘reject x ’ (respectively) when the number x is observed. To link (a) to (b), with random set \mathcal{D} we associate the *coverage function* [23]

$$F(x) := P(x \in \mathcal{D}), \quad x \in \mathbb{R},$$

which defines a kernel. We say that two random sets are *equivalent* if they have the same coverage function. Note that the idea of equivalence class leaves open alternative concepts of a strategy associated with given F .

If Bob can only use integer numbers, then with every $F : \mathbb{Z} \rightarrow [0, 1]$ we can associate a random set by means of a Bernoulli process, that is by including x in \mathcal{D} with probability $F(x)$, independently for all $x \in \mathbb{Z}$. For the game on reals the analogous construction will lead to a set which is not measurable, hence the connection between (a) and (b) is more subtle.

To avoid annoying complications we shall assume that F is *cadlag*, that is right-continuous with left limits. The instance of randomised threshold strategy neatly fits in the random set framework: in this case $\mathcal{D} = [T, \infty)$, and since $x \in \mathcal{D}$ means the same as $T \leq x$ the coverage function is the cumulative distribution function (4) of the random threshold T . To avoid separate treatment of blind-guessing strategies it will be convenient to consider them as special cases of the threshold strategy, with $-\infty$ and ∞ being legitimate values for the threshold. A threshold strategy with $P(T = \pm\infty) = 0$ will be called *proper*, hence satisfying

$$\lim_{x \rightarrow -\infty} F(x) = 0, \quad \lim_{x \rightarrow \infty} F(x) = 1. \quad (9)$$

When Bob plays exchangeable X, Y with some given joint distribution and Alice uses \mathcal{D} with some coverage function F the outcome of the game is assessed as

$$\begin{aligned} P(\text{win}) &= E[\mathbb{1}_{\{X > Y\}} F(X) + \mathbb{1}_{\{Y > X\}} (1 - F(X))] = \\ &= E[\mathbb{1}_{\{X > Y\}} F(X) + 1(Y > X) - \mathbb{1}_{\{Y > X\}} F(X)] = \\ &= \frac{1}{2} + E[\mathbb{1}_{\{X > Y\}} F(X) - \mathbb{1}_{\{X > Y\}} F(Y)] = \\ &= \frac{1}{2} + E[\mathbb{1}_{\{X > Y\}} \{F(X \vee Y) - F(X \wedge Y)\}] = \\ &= \frac{1}{2} + \frac{1}{2} E\{F(X \vee Y) - F(X \wedge Y)\}, \end{aligned} \quad (10)$$

where \vee, \wedge denote maximum and minimum, respectively. The passage to the third line is justified by exchangeability. The last line follows by the following observation: for symmetric function φ from $\mathbb{1}_{\{X > Y\}} + \mathbb{1}_{\{Y > X\}} = 1$ and exchangeability we have

$$E[(\mathbb{1}_{\{X > Y\}} \varphi(X, Y))] = E[\mathbb{1}_{\{Y > X\}} \varphi(X, Y)] = \frac{1}{2} E[\varphi(X, Y)].$$

At a first glance, formula (10) may seem to disagree with (7), as the averaging will apparently involve more complex two-point coverage probabilities

$P(a \in \mathcal{D}, b \in \mathcal{D})$. However, re-writing (7) as

$$\begin{aligned} p(a, b; D) &= \frac{1}{2} + \frac{1}{2} \mathbb{1}_{\{a \in D^c, b \in D\}} - \frac{1}{2} \cdot \mathbb{1}_{\{a \in D, b \in D^c\}} = \\ &= \frac{1}{2} + \frac{1}{2} \cdot [\mathbb{1}_{\{b \in D\}} - \mathbb{1}_{\{a \in D, b \in D\}}] - \frac{1}{2} [\mathbb{1}_{\{a \in D\}} - \mathbb{1}_{\{a \in D, b \in D\}}] = \\ &= \frac{1}{2} + \frac{1}{2} [\mathbb{1}_{\{b \in D\}} - \mathbb{1}_{\{a \in D\}}], \end{aligned}$$

and averaging over the realisations of the random decision set, for the winning probability with \mathcal{D} against pure strategy $a < b$ we obtain

$$P(\text{win}) = \frac{1}{2} + \frac{1}{2}(F(b) - F(a)) \quad (11)$$

in accord with (10). In particular, for threshold strategy (10) becomes (3). By the analogy with the threshold case, we call general \mathcal{D} *proper*, if the coverage function satisfies (9).

It is very fortunate that probability (11) is expressible via such a simple characteristics of \mathcal{D} . It is immediate from the formula that

- (i-a) A mixed strategy \mathcal{D} is minimax if and only if the coverage function F is nondecreasing,
- (i-b) A mixed strategy \mathcal{D} strongly dominates blind-guessing if and only if the coverage function F is strictly increasing.

In particular, every threshold strategy with fully supported T , possibly with atoms at $\pm\infty$ (blind-guessing component), is winning with probability strictly greater than $1/2$. A more interesting example, stepping away from threshold strategies is the following.

EXAMPLE 3.3 Let T be a fully supported random variable with (strictly increasing) cumulative distribution function F , possibly not proper. With T we associate both the threshold strategy, and a dual strategy which accepts x when x is *smaller* than the sample value of T . The decision set for the dual strategy is $(-\infty, T)$, and the coverage function is $1 - F$. For $0 \leq \gamma < 1$ define a (F, γ) -strategy as a mixture of the threshold strategy and its dual with weights γ and $1 - \gamma$. The coverage function for the (F, γ) -strategy is the convex combination

$$F_\gamma(x) = \gamma F(x) + (1 - \gamma)(1 - F(x)).$$

Clearly, the (F, γ) -strategy is not of threshold type because with probability $1 - \gamma$ the decision set is a left-tailed half-line. We have then for $a < b$

$$P(\text{win}) = \frac{1}{2} + \frac{1}{2}(F_\gamma(b) - F_\gamma(a)) = \frac{1}{2} + \frac{2\gamma - 1}{2}(F(b) - F(a)),$$

which exceeds $1/2$ for $\gamma > 1/2$.

We turn to the admissibility of mixed guessing strategies. A simple sufficient condition is

(ii-a) If \mathcal{D} has a proper coverage function F then \mathcal{D} is admissible.

Indeed, suppose F is proper and \tilde{F} is a coverage function for some strategy, such that $\tilde{F}(b) - \tilde{F}(a) \geq F(b) - F(a)$ for all $a < b$. Sending $a \rightarrow -\infty, b \rightarrow \infty$ we get $\tilde{F}(b) - \tilde{F}(a) \rightarrow 1$, which is only possible when \tilde{F} is also proper. Therefore letting $a \rightarrow -\infty$ yields $\tilde{F}(b) \geq F(b)$, and letting $b \rightarrow \infty$ yields $1 - \tilde{F}(a) \geq 1 - F(a)$. Since $a < b$ arbitrary we have $F = \tilde{F}$, hence \tilde{F} cannot dominate F .

Without engaging in fuller analysis of admissibility, as a partial converse to (ii-a) we have

(ii-b) If \mathcal{D} has coverage function F with $\liminf_{x \rightarrow -\infty} F(x) > 0$ or $\limsup_{x \rightarrow \infty} F(x) < 1$ (so, not proper) then \mathcal{D} is dominated.

Indeed, the increments will be only bigger, if we increase F on some halfline $[x, \infty)$ by a constant, or decrease on some $(-\infty, x)$ by a constant.

If F is not proper and nondecreasing (but not constant), we can dominate \mathcal{D} by any $\tilde{\mathcal{D}}$ with coverage function

$$\tilde{F}(x) = \frac{F(x) - c_1}{c_2 - c_1},$$

where $c_1 = \lim_{x \rightarrow -\infty} F(x)$ and $c_2 = \lim_{x \rightarrow \infty} F(x)$. For instance, the (F, γ) -strategy from Example 3.3 is dominated by the threshold strategy with T distributed according to F ; the dominance is strong if T is fully supported.

If the coverage function F is constant, then \mathcal{D} performs like a blind-guessing strategy, hence \mathcal{D} is not admissible. Constant F appears as a coverage function for a *stationary* (translation invariant) random set \mathcal{D} .

EXAMPLE 3.4 Let \mathcal{P} be the random set of atoms of a homogeneous Poisson point process with rate λ . A realisation of the process is a random scatter of isolated points (atoms). Define a decision set as the pointwise sum of sets $\mathcal{D} = \mathcal{P} + [0, 1]$, which is the union of unit intervals having left endpoints in \mathcal{P} . A point x is uncovered by \mathcal{D} with probability $e^{-\lambda}$, as this occurs whenever \mathcal{P} has no atoms in $[x - 1, x]$. The coverage function is therefore constant $F(x) = 1 - e^{-\lambda}$, hence this strategy always wins with probability $1/2$.

Instead of building upon a Poisson process, one can use any stationary renewal point process, expressing F via the distribution of the age (also known as backward excess) random variable.

Every cadlag nondecreasing F with values in $[0, 1]$ is a distribution function for some random variable, hence the coverage function for a threshold strategy. Putting this together with (i-b), (ii-a) and (ii-b) we arrive at a crucial conclusion

- (ii-c) A guessing strategy \mathcal{D} is superminimax if and only if the coverage function F is strictly increasing and proper. In this case \mathcal{D} is equivalent to the threshold strategy with T having the cumulative distribution function F .

We give a construction of superminimax strategies not of the threshold type.

EXAMPLE 3.5 For λ a positive locally integrable function on \mathbb{R} , consider a (inhomogeneous) Poisson point process \mathcal{P} of intensity λ . Define $\mathcal{D} = \mathcal{P} + [0, 1]$. With positive probability the set \mathcal{D} has many connected components, hence this is not a threshold strategy. The coverage function is computed as $F(x) = 1 - e^{-\Lambda(x)}$, where $\Lambda(x) = \int_{x-1}^x \lambda(t)dt$ is the mean number of Poisson atoms in $[x-1, x]$. To get a proper strictly increasing F , it is enough to require that λ be increasing, with limits at $\pm\infty$ being 0 and ∞ , respectively. To meet these conditions we can take the intensity function $\lambda(t) = e^t$.

Finally, we comment on the game on \mathbb{Z} . If Bob is restricted to use only integer numbers the above arguments simplify in that there are no longer measurability and continuity concerns. A random set $\mathcal{D} \subset \mathbb{Z}$ can be identified with a 0-1 process on the lattice. The simplest such process is homogeneous Bernoulli, which is equivalent to the blind guessing. Borrowing an idea from the interacting particle systems [22] and the theory of generalised exchangeability [16] a superminimax strategy can be constructed as a q -deformation of the Bernoulli process, as in the following example.

EXAMPLE 3.6 There exists a random bijection $\Sigma : \mathbb{Z} \rightarrow \mathbb{Z}$ with the following property. For every n , and a subset $S \subset \mathbb{Z}$ with $2n+1$ elements, given that the (unordered) set of values of $\Sigma|_{[-n,n]}$ is S , with probability proportional to q^I the bijection $\Sigma|_{[-n,n]} \rightarrow S$ coincides with any of $(2n+1)!$ bijections $\sigma : [-n,n] \rightarrow S$ where I is the number of inversions in σ . We can view Σ as an infinite analogue of the Mallows model for finite random permutations, see [16] for details. The decision set will be defined as $\mathcal{D} = \{j : \Sigma(j) \geq 0\}$. The indicator function of \mathcal{D} may be thought of as configuration of ‘particles’ occupying some positions on the lattice. This configuration is obtained by using Σ to shuffle the initial configuration where particles occupy positions $j \geq 0$ while positions $j < 0$ are the ‘holes’. For $0 < q < 1$ the bijection Σ tends to be increasing, and \mathcal{D} has the decisive property that the pattern ‘hole, particle’ in positions $a < b$ is more likely than ‘particle, hole’; thus \mathcal{D} is a superminimax strategy. The random configuration \mathcal{D} is an equilibrium state (also known as ‘blocking measure’) for asymmetric simple exclusion process [22].

3.6. Repeated game The objective interpretation of probability involves the long-run frequency in a series of identical trials. Thus speaking of

probability of an outcome in the guessing game we actually that the game is played in many rounds, and that each player employs the same randomisation device in every round. In the time perspective, the latter means that the strategies are *stationary*. Considering the guessing game as a *repeated game* [28], we may allow playing nonstationary strategies, by changing randomisation method from round to round. The long run frequency of wins is a natural objective for the guessing player. A solution to repeated game exists in stationary strategies if the basic (stage) zero-sum game has a solution. But in the guessing game Bob has only epsilon-minimax strategies, therefore it perfectly makes sense to use nonstationary strategies for writing the numbers, in each successive round making guessing harder. For instance, Bob will bound the long-run frequency of Alice's wins by *exactly* $1/2$ by employing the uniform distribution on the finite set (6) in round m . Then, in terms of the long run frequency, the strategy of blind guessing cannot be outperformed, and the repeated game has a solution.

3.7. B. Hill's assumption and the invariance principle An ideal strategy for Bob is exchangeable pair X, Y without ties satisfying $P(\pi(X) = 1/2) = 1$. This condition is the $n = 1$ instance of *assumption* A_n introduced by Bruce Hill [17] as a model for nonparametric predictive inference. He used threshold comparison to show that A_1 and then, by induction, A_n is impossible. But Lane and Sudderth [21] showed that A_n can be realised in the setting of finitely additive probability theory, which leads to a kind of unconventional minimax strategy for Bob.

The uniform distribution on (6) with large m can be regarded as approximation to A_1 . It is instructive to represent this strategy as

$$X = B + G_1, \quad Y = B + G_2, \quad (12)$$

where the pair G_1, G_2 is exchangeable, and independent of the *location parameter* B . Using the additive form many other approximations to A_1 , hence good strategies for Bob can be constructed. More insight comes from the statistical Principle of Invariance which asserts existence of *invariant* minimax strategy in decision problem with certain group of symmetries.

EXAMPLE 3.7 Consider the game on \mathbb{Z} with a further constraint that Bob must use two consecutive integers. A pure strategy for Bob amounts to choosing the location parameter $\beta \in \mathbb{Z}$, to play exchangeable $X = \beta + G_1, Y = \beta + G_2$, where G_1, G_2 is the pair 0, 1 arranged in random order. The group \mathbb{Z} acts in this setting by translations and is *admissible* (cf [2], Definition 8.6.2). The latter means that, for all $\beta, x \in \mathbb{Z}$ when Bob plays pure strategy β and Alice plays a strategy which accepts a given x as the larger number when x is observed, the outcome is the same as in the game where Bob plays $\beta + z$ and Alice accepts $x + z$ when $x + z$ is observed. Since \mathbb{Z} acts transitively on the

range of observed variable, any invariant minimax strategy must be constant, hence it is blind guessing. Uniform distribution on $\{-m, \dots, m\}$ (for big m) is Bob's epsilon-minimax strategy, which may be regarded as approximation to the ideal strategy for Bob, the latter being the infinite invariant measure on \mathbb{Z} .

EXAMPLE 3.8 (Example 3.2 continued) Suppose G_1, G_2 in (12) is the pair of numbers 0, 1 arranged in random order, and that B takes values in the group $3\mathbb{Z}$ of integer multiples of 3. Alice's invariant minimax strategy is constant on the orbits $\{3j : j \in \mathbb{Z}\}$ and $\{3j + 1 : j \in \mathbb{Z}\}$, and it is the sure-fire strategy which accepts $x = 3j + 1$ and rejects $x = 3j$.

EXAMPLE 3.9 To approximate A_1 by continuous distributions we can take

$$X = B + U_1, Y = B + U_2, \quad (13)$$

where B is uniformly distributed on $[-m, m]$ and independent of U_1, U_2 , and the variables U_1, U_2 are iid uniform on $[0, 1]$. Similarly to (12), taking m big we achieve that $\pi(X) = 1/2$ with high probability, and this yields a epsilon-minimax strategy for Bob in the game where he controls a continuous location parameter $\beta \in \mathbb{R}$. The value of this game is $1/2$. The advantage of using (13), as compared to discrete distributions, is that X, Y is a mixture of iid sequences, hence can be extended to an infinite sequence $B + U_i$ of exchangeable random variables. A version of the Game of Googol for such location mixtures was studied in [24].

EXAMPLE 3.10 Yet another approximation to A_1 is obtained via the multiplicative form

$$X = AU_1, Y = AU_2, \quad (14)$$

where A is a positive *scaling parameter* independent of uniform iid U_1, U_2 . This can be interpreted in context of the game where Bob only controls the scaling parameter. To have $E|\pi(X) - 1/2|$ small, one can take A with density

$$P(A \in [a, a + da]) = \frac{1}{2 \log m} \frac{da}{a}, \quad m^{-1} < a < m,$$

which may be seen as a finite approximation to the invariant measure on the multiplicative group \mathbb{R}_+ .

Scale mixtures of uniform distributions, based on another approximation to the invariant measure on \mathbb{R}_+ , will play important role in the remaining part of the paper.

4. The two piles game

4.1. Setup We focus next on a game that bears features of asymmetric two-cards game, the Game of Googol with n cards and the game with two sequences from [30]. Bob takes n separate cards and on each writes a different number. The deck is shuffled and dealt face down in two piles of k and $n - k$ cards. The pile sizes are known to both players. Alice collects the pile with k cards, looks at the numbers and guesses if the number largest of all n numbers is in her hand or in the other pile with $n - k$ cards. She wins if her guess is right.

We model Bob's strategy as a joint probability distribution for n random variables Z_1, \dots, Z_n which do not tie, i.e. $P(Z_i = Z_j) = 0$ for $i \neq j$, and are exchangeable. Exchangeability is equivalent to the property that the sequences $\dots, Z_i, Z_{i+1}, \dots$ and $\dots, Z_{i+1}, Z_i, \dots$ with just two neighbours swapped have the same joint distribution.

The numbers z_1, \dots, z_k in Alice's hand will be identified with values of random variables Z_1, \dots, Z_k , and the hidden numbers in the second pile with values of Z_{k+1}, \dots, Z_n . Using \vee for 'maximum' let $X = Z_1 \vee \dots \vee Z_k$ and $Y = Z_{k+1} \vee \dots \vee Z_n$ be the respective maxima in piles. Having observed z_1, \dots, z_k , Alice's dilemma amounts to guessing which of the inequalities $x > y$ and $x < y$ is true for the values x, y of X, Y , respectively.

The setting is similar to the game with two cards, but there are two delicate distinctions. Firstly, Bob determines the joint distribution of X, Y indirectly, via the joint distribution of Z_1, \dots, Z_n . Secondly, Alice has more information when observing Z_1, \dots, Z_k as compared to just learning the maximum of these k variables. Her best-response strategy is to decide that $x > y$ if

$$P(X > Y | Z_1 = z_1, \dots, Z_k = z_k) \geq \frac{1}{2}, \quad (15)$$

where, in general, the conditional probability depends on all z_1, \dots, z_k in a complex way. For tractability reasons it will be useful to find a framework where this probability is expressible in terms of a few summary statistics of the observed numbers.

By exchangeability,

$$P(X > Y) = \frac{k}{n},$$

because each Z_j is equally likely to be the largest of all n numbers. We call the game *symmetric* if the deck is split evenly (n even, $k = n/2$).

If the game is symmetric, X and Y are themselves exchangeable. Like in the two-cards game, a threshold strategy dominates the blind guessing. One might anticipate that the value of the game is still $1/2$, but the tools we used to show this in the two-cards game are not appropriate for $n > 2$.

In the asymmetric game the pile of bigger size is more likely to contain the overall largest number. Under 'blind guessing' we shall understand Alice's strategy which always decides on the larger pile, regardless of the revealed numbers. Since on the average $E[P(X > Y | Z_1, \dots, Z_k)] = k/n \neq 1/2$, we

may hope to find a joint distribution for Z_1, \dots, Z_n , such that the conditional probability is always within ϵ from the constant k/n , hence compares to $1/2$ in the same way as k/n does. This would also imply the same kind of definite relation between X and the conditional median of Y given Z_1, \dots, Z_k . For such Bob's strategy the blind guessing would be Alice's best response.

4.2. The game with independent numbers Suppose Bob plays iid Z_1, \dots, Z_n with uniform distribution on $[0, 1]$, and Alice plays her best-response (Bayesian) strategy. For $x = z_1 \vee \dots \vee z_k$, condition (15) becomes $P(Y < x) \geq 1/2$, which holds exactly for $x \geq \mu$, where threshold μ is the median of Y .

From $P(X < x) = P(Z_1 < x, \dots, Z_k < x) = x^k$ the density function of X is kx^{k-1} for $x \in [0, 1]$. From

$$\frac{1}{2} = P(Y < \mu) = P(Z_{k+1} \leq \mu, \dots, Z_n \leq \mu) = \mu^{n-k}$$

the median is $\mu = 2^{-1/(n-k)}$. The optimal probability of win is computed as

$$\begin{aligned} v &= P(X > Y, X > \mu) + P(X < Y, X < \mu) \\ &= \int_{\mu}^1 x^{n-k} \cdot kx^{k-1} dx + \int_0^{\mu} (1 - x^{n-k}) kx^{k-1} dx \\ &= \frac{k}{n} + \mu^k + \frac{2k}{n} \mu^n, \end{aligned}$$

which upon substituting the median gives

$$v = \frac{k}{n} + \left(1 - \frac{k}{n}\right) 2^{-k/(n-k)}. \quad (16)$$

For symmetric games $k/n = 1/2$, and we arrive at $v = 3/4$ in accord with our previous computation for the two-cards game.

Observe that v is a function of the ratio k/n . This is easy to explain using the invariance of the game under monotonic transformations of the number scale. For increasing function $\varphi : (0, 1) \rightarrow \mathbb{R}$, the change of variables $\varphi(Z_1), \dots, \varphi(Z_n)$ yields an equivalent game, since φ preserves the order relations. Arbitrary continuous distribution for the random variable $\varphi(Z_i)$ can be obtained from uniformly distributed Z_i . In the iid case, the game with pile sizes $\ell k, \ell(n-k)$ is reducible to the game with pile sizes $k, n-k$ by passing to n iid variables $Z'_i = Z_{\ell(i-1)+1} \vee \dots \vee Z_{\ell i}$ ($i = 1, \dots, n$) and noting that $Z_1 \vee \dots \vee Z_{\ell k} = Z'_1 \vee \dots \vee Z'_k$ and similarly for the other pile.

Minimising v as function of k/n , the ratio worst for Alice is about 0.587, with minimum v being close to 0.741. On the other hand, the maximum 1 is achieved for $k/n \rightarrow 0$ or 1.

4.3. A class of multivariate densities Multivariate distributions well suited for our purposes have densities the form

$$f_n(z_1, \dots, z_n) = g_n(z_1 \vee \dots \vee z_n), \quad (17)$$

where $z_j > 0$ for $j = 1, \dots, n$ and g_n is a nonnegative function. Random variables Z_1, \dots, Z_n with joint density f_n are exchangeable because the maximum $z_1 \vee \dots \vee z_n$ is a symmetric function. The total probability integral must be one, therefore g_n must satisfy

$$\begin{aligned} \int_0^\infty \dots \int_0^\infty f_n(z_1, \dots, z_n) dz_1 \dots dz_n \\ &= n \int_0^\infty g_n(z_1) dz_1 \int_0^{z_1} \dots \int_0^{z_1} dz_2 \dots dz_n \\ &= n \int_0^\infty g_n(t) t^{n-1} dt = 1. \end{aligned} \quad (18)$$

For $k < n$ the joint density of Z_1, \dots, Z_k is obtained by integrating out the variables z_{k+1}, \dots, z_n as

$$\begin{aligned} f_k(z_1, \dots, z_k) &= \int_0^\infty \dots \int_0^\infty g_n(x \vee z_{k+1} \vee \dots \vee z_n) dz_{k+1} \dots dz_n \\ &= x^{n-k} g_n(x) + (n-k) \int_x^\infty y^{n-k-1} g_n(y) dy \\ &=: g_k(x), \end{aligned} \quad (19)$$

where g_k , similarly to g_n , is a function of maximum $x = z_1 \vee \dots \vee z_k$. The conditional joint density of Z_{k+1}, \dots, Z_n at locations z_{k+1}, \dots, z_n given the observed numbers $Z_1 = z_1, \dots, Z_k = z_k$ becomes $g_n(x \vee y)/g_k(x)$, where $y = z_{k+1} \vee \dots \vee z_n$. Integrating out z_{k+1}, \dots, z_n , with each variable running from 0 to x , we arrive at

$$\begin{aligned} P(X > Y | Z_1 = z_1, \dots, Z_k = z_k) &= P(Y < x | X = x) \\ &= \frac{x^{n-k} g_n(x)}{g_k(x)} =: \pi_{n,k}(x). \end{aligned} \quad (20)$$

The rationale behind (17) can be now appreciated. The conditional density of Y given $Z_1 = z_1, \dots, Z_k = z_k$ depends only on the maximum x of the observed numbers. In this sense we are back to the two-cards problem with certain class of bivariate distributions for X, Y .

4.4. Scale mixtures of uniform distributions We want to find a function g_n for which $\pi_{n,k}(x)$ is close to the constant k/n for all $x > 0$. An ideal candidate is $g_n(x) = nx^{-n}$ as (20) gives $\pi_{n,k}(x) = k/n$ exactly. But this function cannot be used to define a proper density (17) since the integral in (18) diverges. In what follows we will find a proper deformation of the power function.

An important subclass of multivariate densities (17) is the family of scale mixtures of uniform distributions, which we now introduce. Let U_1, U_2, \dots be iid uniform on $[0, 1]$, and let A be a positive random variable, independent of U_j 's. Consider an infinite exchangeable sequence

$$Z_j = AU_j, \quad j = 1, 2, \dots \quad (21)$$

Suppose A has density h on the positive halfline. Given $A = a$, the variables Z_1, \dots, Z_n are iid, uniform on $[0, a]$, hence their (unconditional) joint density is of the form (17) with

$$g_n(x) = \int_x^\infty a^{-n} h(a) da, \quad n = 1, 2, \dots \quad (22)$$

It is not hard to check that this formula agrees with (19) for $k < n$. Conversely, if g_n is a nonnegative function satisfying (18) and $g'_n \leq 0$ then f_n is the n -dimensional density for (21) with variable A having density $h(a) = -g'_n(a)a^n$, $a > 0$. See more in [13] on this connection.

Observe that for the improper distribution with density $h(a) = a^{-1}$, $a > 0$ formula (22) yields the ideal function $g_n(x) = nx^{-n}$.

To achieve convergence we take a piecewise smooth density

$$h(a) = \begin{cases} c a^{-\delta-1}, & \text{for } a \geq 1, \\ c a^{\delta-1}, & \text{for } 0 < a < 1, \end{cases} \quad (23)$$

with $0 < \delta < 1$ yet to be chosen and $c = 2\delta^{-1}$. Then the integral (22) is easily calculated as

$$g_n(x) = \begin{cases} \frac{c}{n+\delta} x^{-n-\delta}, & \text{for } x \geq 1, \\ \frac{c}{n-\delta} x^{-n+\delta} - \frac{2c\delta}{n^2-\delta^2}, & \text{for } 0 < x < 1, \end{cases} \quad (24)$$

where for the second part we used the decomposition

$$g_n(x) = \int_x^1 a^{-n} (ca^{\delta-1}) da + g_n(1).$$

Now a part of (20) is straight from the first part of (24)

$$\pi_{n,k}(x) = \frac{k+\delta}{n+\delta}, \quad \text{for } x \geq 1, \quad (25)$$

and similarly with slightly more effort the other part becomes

$$\pi_{n,k}(x) = \frac{k-\delta \left(1 + \frac{2(k-\delta)}{n+\delta} x^{k-\delta}\right)}{n-\delta \left(1 + \frac{2(n-\delta)}{k+\delta} x^{k-\delta}\right)}, \quad \text{for } 0 < x < 1. \quad (26)$$

From (25) and (26) we see that for every integer N and given positive $\epsilon < 1/N$ it is possible to choose $\delta > 0$ so small that

$$\left| \pi_{n,k}(x) - \frac{k}{n} \right| < \epsilon$$

holds uniformly in $x > 0$ and $1 \leq k \leq n \leq N$. With this choice of parameter $\pi_{n,k}$ relates to $1/2$ exactly as k/n does, unless $k = n/2$. But then the distribution of Z_1, \dots, Z_n defined by (17) and (24) has the properties that

- (i) for $k > n/2$ the optimal Alice's counter-strategy is to always accept the maximum of the first pile x ,
- (ii) for $k < n/2$ the optimal Alice's counter-strategy is to always reject x ,
- (iii) for $k = n/2$ the optimal Alice's counter-strategy wins with probability at most $1/2 + \epsilon$.

It follows that the value of the two-pile game is $\max(k/n, 1 - k/n)$, and that a blind guessing is the minimax strategy in asymmetric game⁴.

To practically implement the strategy defined by (17), (24) Bob needs to simulate a sample value a (possibly with the aid of techniques from [9]), use a standard uniform random numbers generator for n values u_i , and set $z_i = au_i$. To play the best-response strategy in the Bayesian game Alice may calculate the posterior distribution of the unknown scale parameter using the observed $x = z_1 \vee \dots \vee z_k$, but this update cannot help improving upon the blind guessing in the asymmetric game, and will give only a minor advantage in the symmetric one.

REFERENCES

- [1] B. Berezovsky and A. Gnedin, *The best choice problem*, Moscow, Nauka, 1984. [MR 0768372](#)
- [2] D. Blackwell (1951), On the translation parameter problem for discrete variables, *Annals of Mathematical Statistics* 22: 391–399. [doi: 10.1214/aoms/1177729585](#); [MR 0043418](#)
- [3] D. Blackwell and M.A. Girshick, *Theory of games and statistical decisions*, Dover, N.Y., 1979.
- [4] F.T. Bruss (1996), The fallacy of the two-envelopes problem, *Mathematical Scientist* 21: 112–119.
- [5] F.T. Bruss (2003), Playing a trick on uncertainty, *Newsletter of the European Mathematical Society*, issue 50, pp. 5 and 7–8 (online).
- [6] G. Campbell and S.M. Samuels (1981), Choosing the best of the current crop, *Advances in Applied Probability* 13: 510–532. [MR 0615949](#)
- [7] Y.S. Chow and H. Teicher, *Probability theory: Independence, interchangeability, martingales*, Springer Texts in Statistics, Springer, New York, 1997. [doi: 10.1007/978-1-4612-1950-7](#); [MR 1476912](#)
- [8] T. Cover (1987), Pick the largest number, In: *Open Problems in Communication and Computation*, T. Cover and B. Gopinath eds, Springer, p. 152. [doi: 10.1007/978-1-4612-4808-8](#)
- [9] L. Devroye, *Non-uniform random variate generation*, Springer-Verlag, New York, 1986. [doi: 10.1007/978-1-4613-8643-8](#); [MR 836973](#)
- [10] T. Ferguson (1989), Who solved the secretary problem? *Statistical Science* 4: 282–289. [MR 1015277](#)
- [11] A. Gnedin (1994), A solution to the game of googol, *Annals of Probability* 22:1588–1595. [MR 1303655](#)
- [12] A. Gnedin (1996), Das Spiel Googol, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*. [MR 1422387](#)

⁴A simpler function $g_n(x) = (\delta/2n)x^{-n \mp \delta}$ could have been used (with the same break-point as in (24)), but this leads to $h_n(a) = \frac{\delta}{2}(1 \pm \frac{\delta}{n})a^{\mp \delta - 1}$ depending on n , hence losing universality of the above solution.

- [13] A. Gnedin (1995), On a class of exchangeable sequences, *Statistics and Probability Letters* 25: 351-356. [MR 1363235](#)
- [14] A. Gnedin and U. Krengel (1995), A stochastic game of optimal stopping and order selection, *Annals of Applied Probability* 5: 310-321. [MR 1325055](#)
- [15] A. Gnedin and U. Krengel (1996), Optimal selection problems based on exchangeable trials, *Annals of Applied Probability* 6: 862-882. [MR 1410118](#)
- [16] A. Gnedin and G. Olshanski (2012), The two-sided infinite extension of the Mallows model for random permutations, *Advances in Applied Mathematics* 48: 615-639. [MR 2920835](#)
- [17] B.M. Hill (1968), Posterior distribution of percentiles: Bayes' theorem for sampling from a population, *Journal of the American Statistical Association* 63: 677-691. [MR 0238430](#)
- [18] T.P. Hill (2009), Knowing when to stop, *American Scientist* 97, No.2 (online).
- [19] T.P. Hill and U. Krengel (1992), On the game of googol, *International Journal of Game Theory* 21: 151-160. [MR 1203008](#)
- [20] H.P. Kirschner (1976), On the risk-equivalence of two methods of randomization in statistics, *Journal of Multivariate Analysis* 6: 159-166.
- [21] D. A. Lane and W.D. Sudderth (1978), Diffuse models for sampling and predictive inference, *Annals of Statistics* 6: 1318-1336. [MR 0523766](#)
- [22] T.M. Liggett, *Interacting particle systems*, Springer, 2005.
- [23] I. Molchanov, *Theory of random sets*, Springer, 2005.
- [24] J. Petrucci (1980), On a best choice problem with partial information, *Annals of Statistics* 8: 1171-1174. [MR 585717](#)
- [25] D. Samet, I. Samet and D. Schmeidler (2004), One observation behind two-envelope problems, *American Mathematical Monthly* 111: 347-351. [MR 2057189](#)
- [26] S.M. Samuels (1989), Who will solve the secretary problem? [Comment on Ferguson] *Statistical Science* 4: 289-291.
- [27] S.M. Samuels, *Secretary problems*, In: Handbook of Sequential Analysis, B.K. Ghosh and P.K. Sen eds, Marcel Dekker, NY, 1991. ISBN: 0-8247-8408-1; [MR 1174296](#)
- [28] S. Sorin, *A first course on zero-sum repeated games*, Springer, 2002.
- [29] T.J. Stewart (1978), Optimal selection from a random sequence with learning of the underlying distribution, *Journal of the American Statistical Association*, 73: 775-780. [MR 0521326](#)
- [30] K. Szajowski (1984), Optimal stopping of a sequence of maxima over an unobservable sequence of maxima, *Applicationes Mathematicae* 18: 359-374. [MR 0757886](#)
- [31] P. Winkler (2001), Games people don't play, In: *Puzzlers Tribute: A Feast for the Mind*, D. Wolfe and T. Rodgers eds., AK Peters. ISBN10 : 1-56881121-7

Podaj większą liczbę

Alexander V. Gnedin

Streszczenie Przedmiotem rozważań są odmiany gry o sumie zerowej, gdy Bob wybiera dwa różne numery, a Alice dowiaduje się jedną z nich, by zgadnąć, która z liczb jest większa.

2010 Klasyfikacja tematyczna AMS (2010): Pierwotna: 91A60; Wtórna: 60G09; 60G40; 6L15.

Słowa kluczowe: gra o sumie zerowej, dominacja, problem sekretarki.



Alexander Gnedin is professor at Queen Mary, University of London. He received his Ph.D. in mathematics in Moscow and has been holding academic positions in Russia, Germany, Netherlands and UK. His main research interests are in probability theory, especially combinatorial stochastic processes, random symmetries and sequential stochastic optimisation. References to his research papers are found in MathSciNet under [ID:220636](#)

ALEXANDER GNEDIN
QUEEN MARY UNIVERSITY OF LONDON
SCHOOL OF MATHEMATICAL SCIENCES
MILE END ROAD, LONDON E1 4NS, U.K.
E-mail: a.gnedin@qmul.ac.uk

Communicated by: Krzysztof Szajowski

(Received: 11th of May 20016; revised: 26th of July 2016)
